VACUUM POLARIZATION RENORMALIZATION AND THE GEOMETRIC PHASE

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ABSTRACT: As an application of the renormalization method introduced by the second author we give a causal definition of the phase of the quantum scattering matrix for fermions in external Yang-Mills potentials. The phase is defined using parallel transport along the path of time evolution operators. The renormalized time evolution operators are elements of the restricted unitary group U_{res} of Pressley and Segal. The central extension of U_{res} plays a central role.

For Dirac fermions in (non second quantized) external Yang-Mills fields the 1-particle scattering matrix S is a well-defined unitary operator and moreover it has a canonical second quantization $\hat{S}$ operating in the fermionic Fock space [1]; for twice differentiable vector potentials one only needs to assume certain fall-off properties at spatial and time infinity.

On the other hand, if one tries to expand the quantum scattering matrix in ordinary Dyson-Feynman perturbation series one meets the well-known vacuum polarization divergences, which must be taken care by suitable (infinite) subtractions. Actually, it is only the phase of the vacuum expectation value $< 0|\hat{S}|0 >$ which is diverging; the absolute value is uniquely defined and finite by canonical quantization. The crucial point is that when passing from S to $\hat{S}$ using the rules of canonical hamiltonian quantization the phase remains ill-defined. It is exactly this quantity which is diverging in perturbation theory.

The 1-particle scattering operator S is defined as the limit of the time evolution operator in the interaction picture, $U_I(t,-t) \to S$ as $t \to \infty$. The time evolution in the Schrödinger picture is defined by

$$i\partial_t U(t,t_0) = h(t)U(t,t_0) \text{ with } U(t_0,t_0) = 1, \tag{1}$$

and in the interaction picture by

$$i\partial_t U_I(t,t_0) = V_I(t)U_I(t,t_0), \text{ with } U_I(t_0,t_0) = 1. \tag{2}$$

The interaction is $V_I(t) = e^{ith_0}V(t)e^{-ith_0}$, the total hamiltonian being $h(t) = D_A = h_0 + V(t)$ with $h_0 = D_0 = -i\gamma_0\gamma^k\partial_k$. The quantum divergences are related to the fact that when $V = \gamma^0\gamma^k A_k(\mathbf{x},t) + A_0(\mathbf{x},t)$ is the interaction with a Yang-Mills potential then the quantization of $\hat{U}_I(t,-\infty)$ for intermediate times $t < \infty$ is not

well-defined. For the sake of simplified notation we put the fermion mass $m = 0$. It does not play any role in the treatment of ultraviolet problems below.

The starting point for our discussion here is the following renormalization which makes U_I quantizable, [2,3]. For each (smooth) potential $A = A_0 dt + A_k dx^k$ one defines a unitary operator T_A in the 1-particle space with the following property. Let $\epsilon = h_0/|h_0|$ be the sign of the free hamiltonian. Define $U_{ren}(t, t_0) = T_A(t) U_I(t, t_0) T_A(t_0)^{-1}$. Then 1) $[\epsilon, U_{ren}(t, t_0)]$ is Hilbert-Schmidt, 2) $T_A(t) \to 1$ as $t \to \pm\infty$. The last property quarantees that the renormalization does not affect the scattering matrix S whereas the first condition guarantees that the renormalized time evolution is quantizable in the free Fock space.

The Hilbert-Schmidt condition on operators $[\epsilon, g]$ defines the restricted unitary group U_1 of unitary operators g, [8]. The second quantization of elements in U_1 defines a central extension $\hat{U}_1$ as discussed in detail in [8]. Now we have a smooth path of operators $g(t) = U_{ren}(t, -\infty)$ in U_1 with the initial condition $g(-\infty) = 1$ and $g(+\infty) = S$. The central extension of U_1 defines a natural connection in the circle bundle $\hat{U}_1 \to U_1$. The curvature of this connection has a simple formula,

$$\omega(X, Y) = \frac{1}{4} \mathrm{tr}\, \epsilon [\epsilon, X][\epsilon, Y] \tag{3}$$

where X, Y are tangent vectors at $g \in U_1$, identified as elements of the Lie algebra of U_1. The phase of $\hat{S}$ is defined *through parallel transport,* with respect to the connection above.

Our definition of phase of $\hat{S}$ is *causal:* A scattering process in an external field A followed by the scattering in A' defines the same phase as the scattering for $A'' = A \cup A'$, when both A, A' have finite nonoverlapping support in time and A'' denotes a field $A''(t) = A(t)$ for $t < t_0$ and $A''(t) = A'(t)$ for $t > t_0$; here t_0 separates the supports of A, A'. The operator T_A is not uniquely defined. It is more convenient to define the transformation first in the Schrödinger picture (1). A simple formula which works in the gauge $A_0 = 0$ is (here $\not{E} = \partial_t \not{A}$)

$$T_A = \exp\left(\frac{1}{4} [\frac{1}{D_0}, \not{A}] - \frac{1}{8} [\frac{1}{D_0} \not{A} \frac{1}{D_0}, \not{A}] - \frac{i}{4} \frac{1}{D_0} \not{E} \frac{1}{D_0} \right), \tag{4}$$

where it is understood that the singularity at the zero modes of D_0 is taken care of by an infrared regularization, for example $\frac{1}{D_0} \to \frac{D_0}{D_0^2 + \alpha^2}$ for some nonzero real number α. In the interaction picture one uses the operator $\exp(ith_0) T_A \exp(-ith_0)$.

The validity of the choice (4) is proven as follows. The time evolution equation for $U_{ren}(t, -\infty) = T_A U(t, -\infty)$ is

$$i\partial_t U_{ren}(t) = (h_0 + W(t)) U_{ren}(t)$$

with

$$W(t) = (i\partial_t T_A) T_A^{-1} + T_A (h + V(t)) T_A^{-1}.$$

Expanding the exponential and arranging terms according to powers of the inverse of momentum (i.e., of D_0,) one gets $[\epsilon, W] = R_1 + R_2 + \ldots$, where the dots denote terms which behave explicitly as $|D_0|^k$ with $k \leq -2$ for high momenta, and

$$R_1 = \frac{1}{2} \frac{D_0}{|D_0|} \not{A} - \frac{1}{2} \not{A} \frac{D_0}{|D_0|} + \frac{1}{4} |D_0| \not{A} \frac{1}{D_0} - \frac{1}{4} D_0 \not{A} \frac{1}{|D_0|} + \frac{1}{4} \frac{1}{|D_0|} \not{A} D_0 - \frac{1}{4} \frac{1}{D_0} \not{A} |D_0|$$

and the second term, which is quadratic in $\not{A}$, is

$$R_2 = \frac{1}{4}\frac{1}{D_0}[|D_0|, \not{A}^2]\frac{1}{D_0} + \frac{1}{4}\left[\frac{1}{|D_0|}, \not{A}^2\right] + \frac{1}{8}D_0\left[\not{A}\frac{1}{D_0}\not{A}, \frac{1}{|D_0|}\right] + \\ + \frac{1}{8}\left[\not{A}\frac{1}{D_0}\not{A}, \frac{1}{|D_0|}\right]D_0 + \frac{1}{8}\frac{1}{D_0}\left[\not{A}\frac{1}{D_0}\not{A}, |D_0|\right] + \frac{1}{8}\left[\frac{1}{D_0}\not{A}\frac{1}{D_0}, |D_0|\right]\frac{1}{D_0}.$$

Since the commutator $[|D_0|^k, \not{A}]$ is of order $|D_0|^{k-1}$ in momenta, all terms in R_2 are actually of order -2 or less. In order to estimate R_1 we write it in equivalent form

$$R_1 = \frac{1}{4}\frac{D_0}{|D_0|}\left[[\not{A}, |D_0|], \frac{1}{|D_0|}\right] - \frac{1}{4}\left[[\not{A}, |D_0|], \frac{1}{|D_0|}\right]\frac{D_0}{|D_0|}$$

and observe both terms are of order -2 for the same reason as in the case of R_2. We have disregarded all the low order terms because in three space dimensions (in a box) the condition that a PSDO is Hilbert-Schmidt (which was required for canonical quantization) is precisely the requirement that the operator vanishes for high momenta faster than $|p| = |D_0|$ raised to power $-3/2$. Thus after our renormalization (= conjugation by the time-dependent unitary operator T_A) the gauge interaction can be lifted to a finite operator in the fermionic Fock space.

This method can be extended to other interactions as well, under very general conditions [3]. It has been also used to derive the chiral anomaly in the hamiltonian framework [2], [3].

The curvature formula (3) is *nonlocal*. The computation of the trace involves space derivatives up to arbitrary high power because of the Green's function $1/D_0$ in the renormalization prescription. However, the curvature is equivalent in cohomology to a local formula. For bounded pseudodifferential operators satisfying the condition that the degree of the commutator $[\epsilon, X]$ is less than $-n/2$ (here the space dimension $n = 3$) we have

$$\omega_{loc}(X, Y) = \mathrm{Res}\,\epsilon[\log|p|, X]Y, \tag{5}$$

where $|p|$, the length of three momentum, is the symbol of $|D_0|$. The difference $\omega - \omega_{loc}$ is a coboundary, [2], [4]. Here we have used the operator residue [7] for PSDO's. Without the sign ϵ this formula would give the Radul cocycle, which is a 2-cocycle on the algebra of *all* PSDO's on a compact manifold [5]. In quantum field theory it is important to keep the sign because this is the cocycle arising from normal ordering. Let us recall how the normal ordering is related to the curvature ω and Schwinger terms.

We want to concentrate on the ultraviolet behavior of operators and therefore we put the system in a box (or assume from very beginning that the Dirac particle is moving on a compact manifold). Let a_i^*, a_j, with $i, j \in \mathbb{Z}$, be a complete set of creation and annihilation operators such that the index $i \geq 0$ corresponds to nonnegative energies and $i < 0$ to negative energies,

$$a_i^* a_j + a_j a_i^* = \delta_{ij}.$$

The normal ordering is defined by $:a_i^* a_j: = a_i^* a_j$ except when $i = j < 0$ and then $:a_i^* a_j: = -a_j a_i^*$. If X, Y are a pair of one-particle operators then the canonical quantizations

$$\hat{X} = \sum X_{ij} :a_i^* a_j:$$

satisfy the algebra

$$[\hat{X},\hat{Y}] = \widehat{[X,Y]} + \omega(X,Y),$$

and the quantum operators are defined such that $[\hat{X}, a_i^*] = \sum X_{ji} a_j^*$. The equivalence of the two cocycles means simply that if $\tilde{X} = \hat{X} + \lambda(X)$ for a suitable complex linear form on the algebra of one-particle operators then

$$[\tilde{X},\tilde{Y}] = \widetilde{[X,Y]} + \omega_{loc}(X,Y). \tag{6}$$

It is also interesting to note that the local 2-cocycle can be written as

$$\omega_{loc}(X,Y) = \mathrm{Res}\frac{1}{|D_0|}[D_0,X]Y. \tag{7}$$

This is one of the cyclic cocyles in noncommutative geometry [6]. (But remember that in three space dimensions we must impose the restrictions mentioned earlier on the PSDO's X, Y.)

The great advantage of the residue formula is that it only depends on the term in the appropriate PSDO which is precisely of asymptotic degree $-n$ (here -3) when expanded asymptotically in powers of $1/|p|$. In operator products each derivative in configuration space is associated with a derivative in momentum space. Since a differentiation in momentum space decreases the homogeneous degree of the operator it follows that a calculation of the residue can involve derivatives with respect to x_i's only up to a finite order.

We finally comment on the renormalization dependence of the phase of the quantum scattering matrix $\hat{S}$. Suppose that we have constructed another renormalization operator T'_A such that each term in the asymptotic expansion

$$T'_A = 1 + t_{-1}(A) + t_{-2}(A) + \ldots,$$

in homogeneous terms t_k of degree k in momenta, is a local differential polynomial in the components of the vector potential A. Then the change of phase of the quantum scattering is given by a parallel transport around a loop of time evolution operators obtained by first travelling the path $g(t)$ from the point $g(-\infty) = 1$ to $g(\infty) = S$ defined by the renormalized time evolution due to T_A and then following backwards in time the time evolution defined by the renormalization T'_A. But the logarithm of the parallel transport phase is equal to the integral of the curvature over a surface bounded by the loop. Since the curvature formula is local, the value of the integral is given by a space integral of a local differential expression in the vector potential. This is in the spirit of local quantum field theory: A change in the renormalization corresponds to a local counterterm in the Lagrangian.

Generically, a given choice of T_A does not lead to a gauge invariant phase even in the case of Dirac fermions. (For Weyl fermions this would be expected because of the chiral anomaly.) But this is not a serious problem and it can be avoided by introducing local counterterms to the hamiltonian as will be discussed in detail in [9].

References

[1] J. Palmer, J. Math. Anal. Appl **64,** 189 (1978). S.N.M Ruijsenaars, J. Math Phys. **18,** 720 (1976) and J. Funct. Anal. **33,** 47 (1979)

[2] J. Mickelsson, In: *Integrable Systems and Strings.* Ed. by A. Alekseev et al. Springer Lecture Notes in Physics **436** (1994)

[3] E. Langmann and J. Mickelsson, J. Math. Phys. **37,** 3933 (1996)

[4] M. Cederwall, G. Ferretti, B.E.W. Nilsson, A. Westerberg, Commun. Math. Phys. **175,** 221 (1996)

[5] A.O. Radul, Funct. Anal. Appl. **25,** 33 (1991)

[6] A. Connes: *Noncommutative geometry.* Academic Press (1994)

[7] M. Wodzicki, In: *K-theory, arithmetic and geometry.* (ed. by Yu.I. Manin) Lect. Notes in Math. **1289,** 320-399, Springer-Verlag, Berlin, Heidelberg, New York (1987)

[8] A. Pressley and G. Segal: *Loop Groups.* Clarendon Press, Oxford (1986)

[9] J. Mickelsson, in preparation